1) (a) $$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{j=1}^{k} \frac{\alpha_j}{1-jx} = \frac{\alpha_1}{1-x} + \frac{\alpha_2}{1-2x} + \cdots + \frac{\alpha_k}{1-kx}$$

$\times\,(1-rx) \Rightarrow \quad (1 \le r \le k)$

$$\frac{\cancel{(1-rx)}}{(1-x)\cdots\cancel{(1-rx)}\cdots(1-kx)} = \frac{\alpha_1(1-rx)}{1-x} + \cdots \frac{\alpha_r\cancel{(1-rx)}}{\cancel{1-rx}} + \cdots + \frac{\alpha_k(1-rx)}{1-kx}$$

Let $x = \frac{1}{r}$ $\quad$ RHS $= 0 + \cdots + \alpha_r + \cdots + 0 = \dfrac{1}{\underbrace{[(1-\frac{1}{r})\cdots(1-\frac{r-1}{r})][(1-(\frac{r+1}{r}))\cdots(1-\frac{k}{r})]}_{k-1 \text{ factors}}}$

$$\therefore \alpha_r = \frac{r^{k-1}}{[(r-1)(r-2)\cdots(r-(r-1))]\underbrace{[(r-(r+1))\cdots(r-k)]}_{k-r \text{ factors}}}$$

$$= \frac{r^{k-1}(-1)^{k-r}}{(r-1)!\,[(\overline{r+1}-r)(\overline{r+2}-r)\cdots(k-r)]} = \frac{(-1)^{k-r}\,r^{k-1}}{(r-1)!\,(k-r)!}$$

(b) $b(n)$ = # of distinct ways of expressing $\{1,\ldots,n\}$ as the disjoint union of non-empty subsets.

$n = 3$ $\quad \{\{1,2,3\}\}$ $\quad$ are the 5 ways of doing this.

$\{\{1,2\},\{3\}\}$

$\{\{1,3\},\{2\}\}$

$\{\{2,3\},\{1\}\}$

$\{\{1\},\{2\},\{3\}\}$

Given $B(x) = e^{e^x - 1}$ $\qquad B(0) = e^{e^0-1} = e^{1-1} = e^0 = 1$. Let $\mathrm{Exp}(x) = e^x$

$B'(x) = \mathrm{Exp}(e^x + x - 1) \quad \Rightarrow \quad B'(0) = \mathrm{Exp}(e^0 - 1) = 1$

$B''(x) = \mathrm{Exp}(e^x + x - 1)[e^x + 1] \Rightarrow B''(0) = \mathrm{Exp}(e^0 - 1)\,2 = 2$

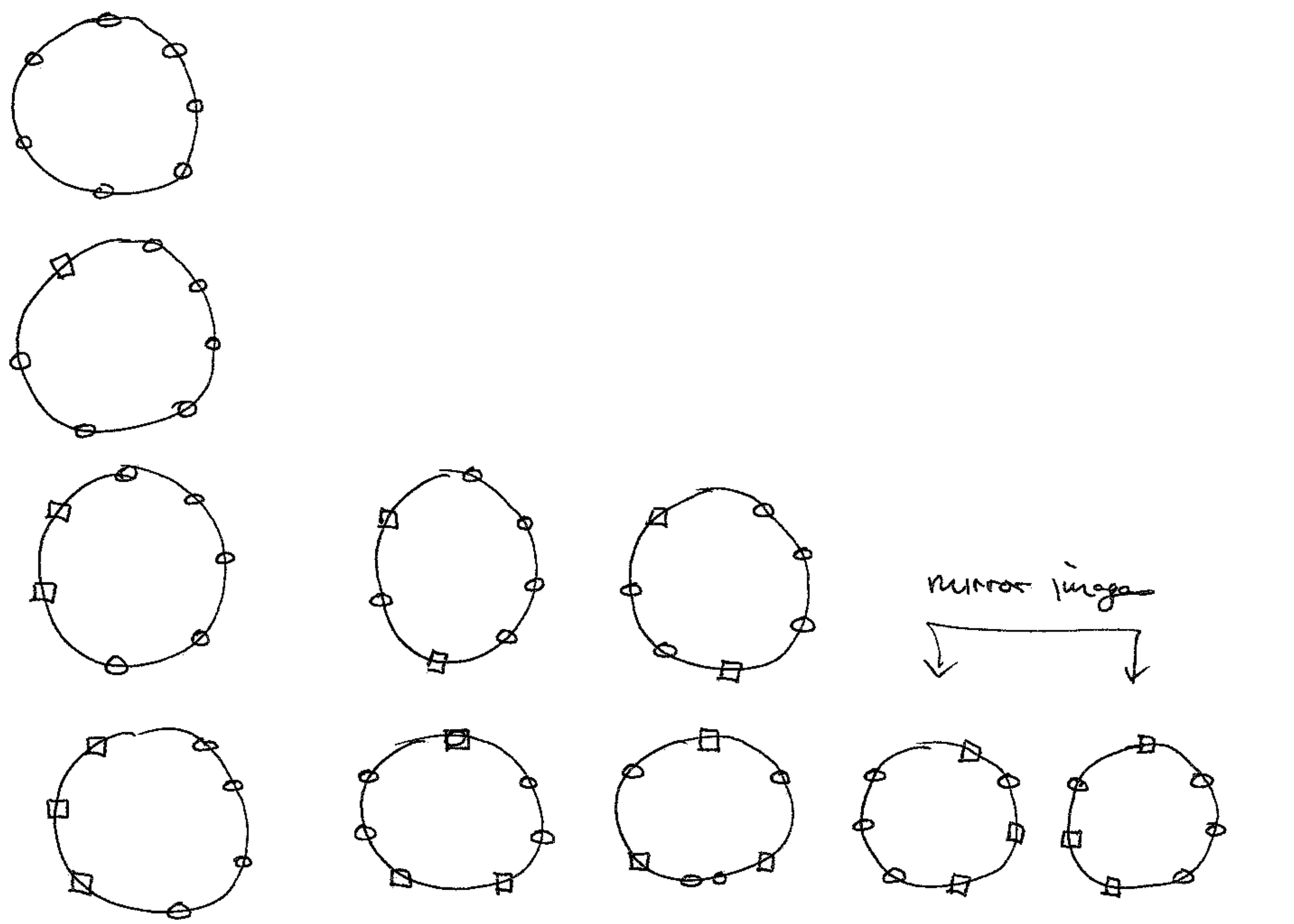
$B'''(x) = \mathrm{Exp}(e^x + x - 1)e^x + [e^x + 1]^2\,\mathrm{Exp}(e^x + x - 1)$

$\Rightarrow \quad B'''(0) = \mathrm{Exp}(e^0 - 1)e^0 + [e^0 + 1]^2\,\mathrm{Exp}(e^0 + 0 - 1)$

$= 1 + 4 \times 1 = 5$ so $\dfrac{B'''(0)}{3!} = \dfrac{5}{6}$

Hence $\quad b(3) = 5$ is confirmed

2.) (a) 7 beads with 2 colours denoted ○ and □ :



- and another 10 necklaces swapping ○ ⟷ □
- note that with the given group mirror image but distinct necklaces are <u>not</u> equivalent.

(b) Burnside – Polya
$$\# = \frac{(p-1)a + a^p}{p} \qquad p = \#\text{beads}. \quad a = \#\text{colours}$$

so here
$$\# = \frac{(7-1)2 + 2^7}{7} = 20.$$

3) (a) <u>Sieve of Erastothenes</u> : Sieve by primes up to $\sqrt{40}$ i.e. $\{2, 3, 5\}$ and not including 1 : retaining the primes we sieve by

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~

multiples of 2 /
multiples of 3 \
multiples of 5 —

$\Longrightarrow \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\} \subset \mathbb{P}$

i.e. <u>12</u> primes up to 40.

(b) $D_2 = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,$
$32,34,36,38,40\}$ $\quad |D_2| = 20 = \lfloor \frac{40}{2} \rfloor$

$D_3 = \{3,6,9,12,15,18,21,24,27,30,33,36,39\}$ $\quad |D_3| = 13 = \lfloor \frac{40}{3} \rfloor$

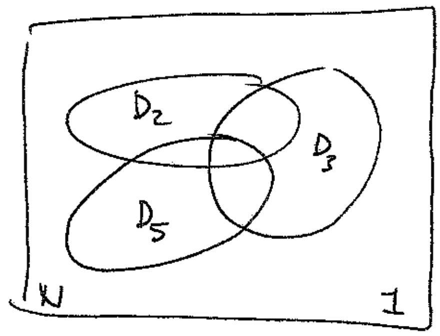
$D_5 = \{5,10,15,20,25,30,35,40\}$ $\quad |D_5| = 8 = \lfloor \frac{40}{5} \rfloor$

$D_2 \cap D_3 = \{6,12,18,24,30,36\}$ $\quad |D_2 \cap D_3| = 6 = \lfloor \frac{40}{6} \rfloor$

$D_3 \cap D_5 = \{15,30\}$ $\quad |D_3 \cap D_5| = 2$

$D_2 \cap D_5 = \{10,20,30,40\}$ $\quad |D_2 \cap D_5| = 4$

$D_2 \cap D_3 \cap D_5 = \{30\} \Rightarrow |D_2 \cap D_3 \cap D_5| = 1.$



$\therefore$ numbers up to 40 not divisible by 2, 3 or 5

give $\quad N - (|D_2| + |D_3| + |D_5|) + (|D_2 \cap D_3| + |D_2 \cap D_5| + |D_3 \cap D_5|)$
$- (|D_2 \cap D_3 \cap D_5|)$ numbers i.e.

$$40 - (20 + 13 + 8) + (6 + 2 + 4) - 1 = 40 - 41 + 12 - 1 = 10$$

These numbers 12 and 10 are consistent since the second number includes 1 but does not include 2, 3 or 5.

(c) # Perms of $\{1,2,3,4,5\}$ fixing 2 elements is given by the coef of x^2 in the expansion

$$E(x) = \sum_t e_t x^t = n! \sum_{r=0}^{n} \frac{(x-1)^r}{r!} \qquad \text{case } n = 5$$

$r = 0 \qquad \frac{5!}{0!}(x-1)^0 = 5! = 120 \qquad \text{coef } x^2 = 0$

$r = 1 \qquad \frac{5!}{1!}(x-1)^1 = 120x - 120 \qquad \text{coef } x^2 = 0$

$r = 2 \qquad \frac{5!}{2!}(x-1)^2 = 5.4.3(x^2 - 2x + 1) \qquad \text{coef } x^2 = 60$

$r = 3 \qquad \frac{5!}{3!}(x-1)^3 = 5.4.(x^3 - 3x^2 + 3x - 1) \qquad \text{coef } x^2 = -60$

$r = 4 \qquad \frac{5!}{4!}(x-1)^4 = 5(x^4 - 4x^3 + 6x^2 - 4x + 1) \qquad \text{coef } x^2 = 30$

$r = 5 \qquad \frac{5!}{5!}(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 \qquad \text{coef } x^2 = -10$

$\therefore$ # perms fixing 2 is $0 + 0 + 60 - 60 + 30 - 10 = 20$