(4) (P04)

$$|\mathcal{L}|_x = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} a_n x^n$$

Binomial Thm
$$(1-4x)^{\frac{1}{2}} = \overset{b_0}{1} + \tfrac{1}{2}\cdot x(-4) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2(-4)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x^3(-4)^3 +$$

$$\ldots \frac{\overbrace{\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\ldots(\tfrac{1}{2}-j+1)}^{j \text{ terms.}}}{1.2.3\ldots j} x^j(-4)^j = \cancel{\Sigma}$$

so
$$b_n = (-1)^n 2^n \frac{(\frac{2}{2})(1-2)(1-2.2)(1-2.3)\ldots(1-2n+2)}{1.\ 2\ \ 3 \ldots\ldots n.}$$

$$= -2^n \frac{1.(2-1)(2.2-1)(2.3-1)\ldots(2n-1-2)}{1.2.3\ldots n.}$$

$$= -\frac{2^n}{n!} \overbrace{1.1.3.5.7\ldots}^{n-1 \text{ terms.}}(2n-3).$$

$$= -\frac{1.2.3.4.5.7.8\ldots(2n-3)(2n-2)\times 2.}{n!(n-1)!}$$

$$b_0 = 1$$

$$= -\frac{2(2n-2)!}{n!(n-1)!}$$

Hence
$$|\mathcal{L}|_x = \frac{1-\sum_{n=0}^{\infty} b_n x^n}{2x} = \frac{1-b_0}{2x} - \frac{\sum_{n=1}^{\infty} b_n x^n}{2x}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} b_n x^{n-1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} b_{n+1} x^n = \sum_{n=0}^{\infty} \frac{(2(n+1)-2)!}{(n+1)!\,n!}x^n.$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!\,n!}x^n$$

Hence
$$a_n = \frac{(2n)!}{(n+1)!\,n!}$$

1 (a) $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=0}^{\infty} (3n+2)x^n = 3\sum_{n=0}^{\infty} n x^n + 2\sum_{n=0}^{\infty} x^n$

$|x|<1$ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n \dots$

$\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots (n+1)x^n.$

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots + nx^n = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n.$

Here. $f(x) = \frac{3x}{(1-x)^2} + \frac{2}{(1-x)} = \frac{3x + 2(1-x)}{(1-x)^2} = \frac{3x + 2 - 2x}{(1-x)^2}$

$= \frac{x+2}{(1-x)^2}.$

(b). $a_n = n^3$. $f(x) = \sum_{n=0}^{\infty} n^3 x^n$

now: $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots nx^n.$

$\Rightarrow \left(\frac{x}{(1-x)^2}\right)' = 1 + 2^2x + 3^2x^2 + n^2x^{n-1} + \dots$

$x\left(\frac{x}{(1-x)^2}\right)' = x + 2^2x^2 + 3^2x^3 + \dots + n^2x^n.$

$\Rightarrow \left(x\left(\frac{x}{(1-x)^2}\right)'\right)' = 1 + 2^3x + 3^3x^2 + \dots n^3x^{n-1} + \dots$

$x\left(x\left(\frac{x}{(1-x)^2}\right)'\right)' = x + 2^3x^2 + 3^3x^3 + \dots n^3x^n + \dots$

$= \frac{x(1+4x+x^2)}{(1-x)^4}$

$\textcircled{2}$ (a) $a_{n+2} = 3a_{n+1} - 2a_n$. $\quad a_0 = 0, a_1 = 1$.

PS2

$$a_0 + a_1 x + \sum_{n=0}^{\infty} a_{n+2} x^{n+2} =: f(x)$$

$$x + x^2 \sum_{n=0}^{\infty} a_{n+2} x^n = x + x^2\left(\sum_{n=0}^{\infty}(3a_{n+1} - 2a_n)x^n\right.$$

$$= x + 3x^2 \sum_{n=0}^{\infty} a_{n+1} x^n - 2x^2 \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{f(x)}$$

$$= x + 3x \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 2x^2 f(x) \checkmark$$

$$\Rightarrow \quad f(x) = x + 3x\left(f(x) - a_0\right) - 2x^2 f(x).$$

So $\quad f(x)(1 - 3x + 2x^2) = x$

$$\Rightarrow \quad f(x) = \frac{x}{2x^2 - 3x + 1} = \frac{x}{(2x-1)(x-1)} = \frac{1}{1-2x} - \frac{1}{1-x}.$$

$$= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n$$

Here.

$$= \sum_{n=0}^{\infty}(2^{n}-1)x_n \quad \Rightarrow \quad a_n = 2^n - 1 \text{ check.}$$

$a_0 = 2^0 - 1 = 0$

$a_1 = 2^1 - 1 = 1$. ✓

(b). $a_{n+1} = a_n + 3n$; $a_0 = 1$, $a_1 = 4$

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1}$$

$$= a_0 + x \sum_{n=0}^{\infty} a_{n+1} x^n = 1 + x \sum_{n=0}^{\infty}(a_n + 3n)x^n$$

$$= 1 + x\sum_{n=0}^{\infty} a_n x^n + 3x \sum_{n=0}^{\infty} n x^n$$

$$= 1 + x f(x) + \frac{3x^2}{(1-x)^2} \quad \text{from 1(a).}$$

$$\Rightarrow \quad f(x)(1-x) = 1 + \frac{3x^2}{(1-x)^2}$$

$$\Rightarrow \quad f(x) = \frac{1}{1-x} + \frac{3x^2}{(1-x)^3} = \frac{(1-x)^2 + 3x^2}{(1-x)^3} = \frac{1 - 2x + x^2 + 3x}{}$$

$$= \frac{(1 - 2x + 4x^2)}{(1-x)^3} \checkmark$$

$$= \frac{3}{(1-x)^3} + \frac{4}{1-x} - \frac{6}{(1-x)^2} \checkmark$$

Hence.

2(b) $\frac{1}{1-x} = 1 + x + x^2 + \cdots 1x^n + \cdots$ ✓

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 \cdots (n+1)x^n + \cdots$ ✓

$\frac{2}{(1-x)^3} = 2.1 + 3.2x + \cdots (n+2)(n+1)x^n + \cdots$ ✓

✓

$\therefore \quad a_n = 4 + \quad - 6(n+1) + \frac{3}{2}(n+2)(n+1).$

<u>Check</u>: $a_0 = 4 - 6 + \frac{3}{2}2.1 = 1.$ ✓

$a_1 = 4 - 12 + \frac{3}{2}3.2 = 1$ ✓

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(3) b(4) is the # of set partitions of a set with 4 elements.

Check. $b(4) = 15$. Let $A = \{1, 2, 3, 4\}$.

$\{\{1\}, \{2,3,4\}\}$
$\{\{1\}, \{2,3\}, \{4\}\}$
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$\{\{1\}, \{2,4\}, \{3\}\}$
$\{\{1,2\}, \{3,4\}\}$
$\{\{1,3\} \{2,4\}\}$
$\{\{1,4\} \{2,3\}\}$
$\{\{2\}, \{1,3,4\}\}$
$\{\{3\}, \{1,2,4\}\}$
$\{\{4\}, \{1,2,3\}\}$
$\{\{1\}, \{2\} \{3,4\}\}$
$\{\{1\} \{4\} \{2,3\}\}$
$\{\{3\} \{4\} \{1,2\}\}$
$\{\{1,2,3,4\}\}$
$\{\{1\}, \{2\}, \{3\}, \{4\}\}$

by hand!
$b(4) = 15.$

Series[Exp[Exp[x] - 1], {x, 0, 5}] $\longrightarrow$

$1 + x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$

so $\frac{b(4)}{4!} = \frac{5}{8}$

$\Rightarrow b(4) = \frac{4 \times 3 \times 2 \times 5}{8} = 15$ //