MATH312 Algebra - Test 2

Answer at least six questions.

1. Describe how you would determine whether or not a subset S of a ring R is a subring and how you would determine if it was an ideal.

Consider the ring of 2×2 matrices with integer coefficients.

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

Verify that the set of upper triangular matrices

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

is a subring of R but is not an ideal.

2. Let $\phi : R \longrightarrow S$ be a ring homomorphism. If I is an ideal of S, prove that $\hat{I} = \{r \in R : \phi(r) \in I\}$ is an ideal of R. (This is called the pullback into R of the ideal I).

3. State the definition of a principal ideal and of a PID (principal ideal domain). Prove that quotients of PIDs domains are PIDs. i.e. show that if R is a PID then $\overline{R} = R/I$ is also a PID for any ideal I of R. You may find the result in the previous question of use.

4. Give an example of a field with 9 elements. Explain fully how you know that your example is a field.

5. Determine whether the following polynomials are irreducible over $\mathbb{Q}$. Explain your reasoning in each case.

a) $3x^3 + 5x + 1$

b) $3x^7 + 6x^3 + 12x + 4$

c) $\frac{1}{2}x^4 - 8$

6. Explain fully the relationships proved in class between the following types of rings.

- Noetherian Ring
- UFD
- Euclidean Domain
- PID
- Ring in which irreducibles are prime.

You may wish to illustrate the relationship with a diagram.

7. Is $\mathbb{Z}_5[x]$ a UFD? Explain, stating any relevant results. How does the factorisation

$$3x^2 + 4x + 3 = (x + 4)(3x + 2) = (2x + 3)(4x + 1)$$

relate to your answer.

8. Show that $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorisation domain by considering possible factorisations of 10 in this ring. The complex norm may be useful here.

9. Consider the ring $F[x]$ where F is a field. Show that the set

$$I = \{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 : a_n + a_{n-a} + \ldots + a_1 + a_0 = 0\}$$

consisting of polynomials in $F[x]$ whose coefficients add to zero, is an ideal in $F[x]$. Is this a principal ideal?

10. Prove directly that every ideal $I \leq \mathbb{Z}$ is equal to $\langle a \rangle = a\mathbb{Z}$ for some element a. This shows that $\mathbb{Z}$ is a PID. This proof was given in class.