(1) If $f(x)$ had a non trivial factor over $\mathbb{Z}_{11}$ it would have a root

so just try $f(a)$ for $a = 0, 1, 2, \ldots, 10$

$f(0) \equiv 4 \mod 11$
$f(1) \equiv 6 \mod 11$
$f(2) \equiv 10 \mod 11$
$f(3) \equiv 5 \mod 11$
$f(4) \equiv 2 \mod 11$
$f(5) \equiv 1 \mod 11$
$f(6) \equiv 2 \mod 11$
$f(7) \equiv 5 \mod 11$
$f(8) \equiv 10 \mod 11$
$f(9) \equiv 6 \mod 11$
$f(10) \equiv 4 \mod 11$

$\Rightarrow$ it is never zero mod 11 $\therefore$ irreducible over $\mathbb{Z}/11\mathbb{Z}$.

If $p = 2$ then $f(x) = x^2 + x + 0 = x(x+1)$

If $p = 3$ then $f(x) = x^2 + x + 1$ and $f(1) = 3 \equiv 0 \mod 3$.

$\therefore$ $x - 1 \equiv x + 2$ is a factor

$$\begin{array}{r} x + 2 \\ x+2 \overline{) x^2 + x + 1} \\ x^2 \; 2x \\ \hline -x \equiv 2x + 1 \\ 2x + 1 \\ \hline \end{array}$$

$\therefore$ $x^2 + x + 4 = (x+2)^2$ over $\mathbb{Z}/3\mathbb{Z}$.

It is irreducible over $\mathbb{Z}$ since it is irreducible over $\mathbb{Z}/11\mathbb{Z}$.

Hence it is irreducible over $\mathbb{Q}$.

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(2) We need to assume R is commutative and has a unity 1.

First AB is an ideal: $x = \sum_{i \in I} a_i b_i$ $y = \sum_{i \in J} a_i b_i$

$$\Rightarrow x + y \in \sum_{i \in I \cup J} a_i b_i \in AB$$

and if $r \in R$, $rx = \sum_{i \in I} (ra_i) b_i$ But $ra_i = a_i' \in A$ since A is an ideal.

$$\Rightarrow rx = \sum_{i \in I} a_i' b_i \in AB.$$

Thus AB is an ideal of R.

Let $x = \sum_{i \in I} a_i b_i \in AB$. Then $\forall i$ $a_i b_i \in A$ since A is an ideal. ideals are closed under $+$ $\therefore$ $\sum_{i \in I} a_i b_i \in A$. Similarly $x \in B$ $\therefore AB \subset A \cap B$.

If. $A + B = R \Rightarrow \exists\, a \in A, b \in B$ such that $1 = a + b$. If $x \in A \cap B$ then write $x = ax + xb \in AB + AB = AB$

$\therefore$ $A \cap B \subset AB \Rightarrow AB = A \cap B$ //

(3) $f(1) = 0$ in $\mathbb{Z}_2$ here. $x-1 = x+1$ is a factor

Divide it out $x+1 \overline{) x^3+x^2+x+1}$ or

note that $f(x) = x^2(x+1) + (x+1) = (x+1)(x^2+1)$

and since $-1 \equiv 1 \bmod 2$. $x^2-1 = x^2+1$

But $x^2-1 = (x+1)(x-1)$

$\therefore f(x) = (x+1)(x+1)(x-1) = (x+1)^2$.

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(4) We need only check these polys over $\mathbb{Z}_4$

(a) irreducible by Eisenstein with $p=3$.

(b) If $x-a$ is a factor then we must have $a/1 \Rightarrow a = \pm 1$.
Since a is a root of $f(x) = x^4+x+1$. But $f(1) = 3 \neq 0$, $f(-1) = 1 \neq 0$.
Thus any non trivial factors must be of degree 2 and we can write
$x^4+x+1 \overset{(i)}{=} (x^2+\alpha x+1)(x^2+\beta x+1)$ or $x^4+x+1 \overset{(ii)}{=} (x^2+\alpha x-1)(x^2+\beta x-1)$

since the factors of 1 are ± 1.

In (i), equating the coef of x : $1 = \alpha+\beta$. } so this is not possible.
of x^3 : $0 = \alpha+\beta$ }

In (ii) x : $1 = -\alpha-\beta$ } again not possible. Thus $f(x)$ is irred over $\mathbb{Z}_4$
x^3 : $0 = \alpha+\beta$ } hence $\mathbb{Q}$.

(c). Let $y = 2x$ so $g(x) = 8x^3-6x+1 = y^3-3y+1 = f(y)$.
Then ~~any any root~~ $f(y)$ is always odd, so it has no root.
Hence $g(x)$ has no root either and is thus irreducible.

(d) $f(x) = x^5+5x^2+1$. The only roots can be ± 1 which both fail.
Here if it has facts they must be of degree 2 and 3. etc.
Consider $f(y-1) = (y-1)^5+5(y-1)^2+1 = y^5-5y^4+10y^3-5y^2-5y+5$
which is irreducible by Eisenstein with $p=5$. Then $f(x)$ is irreducible

(5) $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$

$10 = 2.5 = (2 + \sqrt{-6})(2 - \sqrt{-6})$.

$N(2) = 2^2 = 4$. If 2 was reducible, $2 = xy$ then
$4 = N(x)N(y)$ & $N(x), N(y) \neq \pm 1 \Rightarrow N(x) = N(y) = 2$

But if $x = a + b\sqrt{-6}$, $N(x) = a^2 + 6b^2 = 2$ has no sol^n in $\mathbb{Z}$.

$\therefore$ $2 \in R$ is irreducible.

Similarly $N(5) = 5^2 = 25$. & $N(x) = 5 = a^2 + 6b^2$ has no sol^n.

Then none of these 4 factors can be prime. E.g.

$2 \mid 10 = (2 + \sqrt{-6})(2 - \sqrt{-6})$ but $2 \nmid 2 + \sqrt{-6}$ & $2 \nmid 2 - \sqrt{-6}$

All numbers on the right hand side are irreducible:

because both of the numbers on the right hand side are irreducible (like $\mathbb{Z}$).

In R there is only ± 1 as the units (like $\mathbb{Z}$).