University of Waikato
Department of Mathematics
Modern Algebra math310-10A Rings and Fields
Assignment 1

Due Friday 7th May 2010

Hand in your work, clearly labelled with your name, id number and the number of the assignment, through the slot "Modern Algebra/310" outside the Mathematics office.

1. Let R be a ring. Show, using the ring axioms, that for all $a, b, c \in R$, $a(b - c) = ab - ac$.

2. Let $R = \mathbb{Z} + \sqrt{3}\mathbb{Z} = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\} \subset \mathbb{R}$. Prove that, using the operations inherited from the ring $\mathbb{R}$, R is a commutative ring with unity. Show that $2 + \sqrt{3}$ has a multiplicative inverse in R, i.e. is a unit, and find another non-trivial unit.

3. Show that the group of units U of the ring $R = \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\} \subset \mathbb{C}$ is exactly the subset $\{-1, 1, -i, i\}$ by showing each of these is a unit and that any unit must must be in this set.

4. Draw up an addition and a multiplication table for the ring $\mathbb{Z}/4\mathbb{Z}$. Use the latter to find the unit(s).

5. Prove that every subring of $\mathbb{Z}$ has the form $n\mathbb{Z} = \{xn : x \in \mathbb{Z}\}$ for some $n \in \mathbb{N}$. Then show that if $a, b \in \mathbb{N}$, $a\mathbb{Z} \subset b\mathbb{Z}$ if and only if $b \mid a$. Finally by exhibiting an explicit isomorphism (and proving it works) show each of these subrings is isomorphic to $\mathbb{Z}$.

6. The extended Euclidean algorithm gives for each pair of integers a, b with one at least non-zero, another pair of integers u, v such that $ua + vb = (a, b)$, the greatest common divisor. Use the algorithm to find $(a, b), u, v$ when $a = 336$ and $b = 420$.

7. Let $S = \{a + ib : a, b \in \mathbb{Z},\ b \text{ is even}\} \subset \mathbb{Z}[i]$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal.

8. Find all of the maximal ideals in $\mathbb{Z}/8\mathbb{Z}$.

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30th April 2010